Mode Coupling in Space and Time Varying Anisotropic Absorbing Plasmas

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Abstract

A four dimensional systematic mathematical approach for investigating propagation and coupling of wave modes in a slowly varying (in all space directions and time) anisotropic, absorbing plasma is represented. The formalism is especially useful for energy considerations of the waves. It is applicable to general cases of mode conversion in plasmas with general geometries of space-time and magnetic field configurations. A simple example of how this formalism can be applied to practical cases is given.

1 Introduction

Almost all investigations about wave coupling and mode conversion in plasmas were done for cases, where the inhomogeneity of the medium is only in one direction, i.e. plane stratified plasmas (Stix and Swanson 1983, Swanson 1989, Stix 1992 , Budden 1985). The propagation of electromagnetic waves in four-dimensional space and time varying plasmas were studied before (Bernstein & Friedland 1983, Kravtsov 1969) in the context of geometrical optics. But in these studies the coupling of modes was not considered and the mere interest was concentrated an other themes like for example energy relations and the absorption of independent waves. There exist papers about four dimensional mode-conversion (Friedland, Goldner & Kaufman 1987, Kaufman & Friedland 1987), but the mathematical proof represented there is only valid for the very special cases where the coupling point is not a branch point. The reason is, that the different dispersion equations combined there result from one unique dispersion matrix, which is not the case when we consider for example the unified theory of a class of mode conversion problems, represented by Cairns & Lashmore-Davies 1983. Three dimensional transmission of the fast wave around the ion cyclotron resonance were considered by Friedland 1990 using reduction techniques. For recent investigations about combined mode conversion and absorbtion in one dimensional inhomogeneity see Cairns et.al. 1995 and references therein.

If we want to study plasma physics more accurate, the propagation and coupling of electromagnetic waves in media which vary in more than one dimension have many applications. The most important application of it is perhaps in radio-frequency heating and the investigations of plasmas in a tokamak (Cairns 1991). The plasmas there are considered as plane stratified, but being precise, we have also to take into account the curvature of the tokamak. In almost all cases, when wave coupling was under consideration, this curvature was assumed to be negligible, or at most to give only small and unimportant effects. But experience in plasma physics has always shown, that small variations can give huge effects, because generally plasmas are unstable media. But also in the case of stability, it has been shown, that small variations can have essential new results, for example relativistic effects in wave absorptions (Mc Donalds et al 1994). Another point is, that in the papers about mode resonances, the magnetic field in the plasma is variable in only one direction. But we know, that in a okamak, the magnetic field is also toroidal and poloidal. Response functions for a variable toroidal magnetic field were calculated by Catto, Lashmore-Davies and Martin 1993. Another example of the application of wave coupling with general geometries is the ionosphere, which is usually considered as a plane stratified cold plasma (Budden 1985) and is a cost free medium for many plasma wave observations. Calculations taking into account the curvature of the atmosphere could give us new insight in the physics of the ionosphere.

It is the aim of this paper to give a systematic derivation of coupled wave equations in slowly varying plasmas which are three-dimensional inhomogeneous, non stationary, anisotropic and absorbing. In order to be self-contained, we include also a systematic representation of some results already presented on conferences before (Suchy & Sabzevari 1992 a,b) and refer to these papers for the explicit calculations. Some coupled wave equations were also represented there, but they must fulfil special conditions and are not applicable in general, as the derived equations in the present paper are.

2 The Eikonal-Maxwell System of Differential Equations

In a plasma with time and space dispersion the electric current density, dielectric displacement and magnetic induction must be written as integral equations

$$\begin{aligned}
\mathbf{J}(\mathbf{X}) &= \int d^4\mathbf{X}' \sigma^{\text{ker}}(\mathbf{X}-\mathbf{X}';\mathbf{X}) \cdot \mathbf{E}(\mathbf{X}') \\
\mathbf{D}(\mathbf{X}) &= \int d^4\mathbf{X}' \varepsilon^{\text{ker}}(\mathbf{X}-\mathbf{X}';\mathbf{X}) \cdot \mathbf{E}(\mathbf{X}') \\
\mathbf{B}(\mathbf{X}) &= \int d^4\mathbf{X}' \mu^{\text{ker}}(\mathbf{X}-\mathbf{X}';\mathbf{X}) \cdot \mathbf{H}(\mathbf{X}')
\end{aligned} \tag{1}$$

where the tensors of electric conductivity, dielectric - and magnetic permeability are written as the kernels of the integral equations. They have two kinds of dependencies on the space-time four-vectors $\mathbf{X}$ and $\mathbf{X}'$. The ones before the semicolon are fast varying and are due to the collective effects of the medium, and those after the semicolon are slowly varying and due to the features of the medium itself, like density etc. In the integration over time, causality of course must be taken into account, but we redefine it in the kernels, so that the equations can be written in the above compact form, where all integrations are from $-\infty \to +\infty$. These equation must be solved together with the Maxwell equations

$$\begin{aligned}
\nabla \times \mathbf{H}(\mathbf{X}) &= \tfrac{\partial}{\partial t}\mathbf{D}(\mathbf{X}) + \mathbf{j}(\mathbf{X}) \\
\nabla \times \mathbf{E}(\mathbf{X}) &= -\tfrac{\partial}{\partial t}\mathbf{B}(\mathbf{X}).
\end{aligned} \tag{2}$$

To do this, we first transform the integral equation (1) into differential equations with the help of geometrical optics (Suchy & Sabzevari 1992a, Kravstov 1969), by writing $\mathbf{j}, \mathbf{D}$ and $\mathbf{B}$ in eikonal form

$$\begin{aligned}
\mathbf{E}(\mathbf{X}) &= \mathbf{E}(;\mathbf{X}) \exp\{i\phi(\mathbf{X})\} \\
\mathbf{H}(\mathbf{X}) &= \mathbf{H}(;\mathbf{X}) \exp\{i\phi(\mathbf{X})\} \\
\mathbf{j}(\mathbf{X}) &= \mathbf{j}(;\mathbf{X}) \exp\{i\phi(\mathbf{X})\}
\end{aligned} \tag{3}$$

where $\phi(\mathbf{X})$ is the eikonal, which in geometrical optics is a fast varying function of space and time, and $\mathbf{E}(;\mathbf{X}), \mathbf{H}(;\mathbf{X})$ and $\mathbf{j}(;\mathbf{X})$ are the amplitudes, which are slowly varying, because of the

slowness of variation of the medium. The first two equations of (3) can also be considered as the wave entering the plasma, and the last as the movement of the charged particles in reaction of the waves. On principle, constant external fields could be added to (3), but later they give us no new physical results.

Three different kinds of scales are considered here. First, we suppose that on each particle only the particles in the vicinity of that particle in space and time have an essential influence. This scale we define by Δ_1, which is small compared to the size of the plasma in space and time. Hence in (1)

$$\mathbf{X} - \mathbf{X}' = \mathrm{O}(\Delta_1) \ll 1. \tag{4}$$

Then, it is sufficient to integrate only over the vicinity of the point $\mathbf{X}$. The second scale is due to the slowness of the variation of the medium, which we denote by a small number Δ_2. So we can write

$$\frac{\partial \mathbf{E}(;\mathbf{X})}{\partial \mathbf{X}}, \ \frac{\partial \mathbf{H}(;\mathbf{X})}{\partial \mathbf{X}}, \ \frac{\partial \mathbf{G}(;\mathbf{X})}{\partial \mathbf{X}} \ = \ \mathrm{O}(\Delta_2) \ll 1 \tag{5}$$

where $\mathbf{G}(;\mathbf{X})$ is an arbitrary function of the medium. The third scale, which we denote by Λ, is due to the largeness of the frequency and wave number, which are defined as the four dimensional wave vector

$$\mathbf{K}(\mathbf{X}) = \frac{\partial \phi(\mathbf{X})}{\partial \mathbf{X}}. \tag{6}$$

In geometrical optics, this can be written as

$$\mathbf{K}(\mathbf{X}) = \Lambda\, \mathbf{G}(;\mathbf{X}) \qquad \Lambda \gg 1 \tag{7}$$

where $\mathbf{G}(;\mathbf{X})$ is a slowly varying function due to the slowness of variation of the medium and is of order $\mathrm{O}(1)$. From (6) and (7)

$$\phi(\mathbf{X}) = \int^{\mathbf{X}} \mathbf{K}(\mathbf{X}') \cdot d\mathbf{X}' = \Lambda \int^{\mathbf{X}} \mathbf{G}(;\mathbf{X}') \cdot d\mathbf{X}' \approx \mathrm{O}(\Lambda). \tag{8}$$

Hence, because of (5)

$$\frac{\partial}{\partial \mathbf{X}} \mathbf{K}(\mathbf{X}) = \Lambda \frac{\partial}{\partial \mathbf{X}} \mathbf{G}(;\mathbf{X}) = \mathrm{O}(\Lambda \Delta_2). \tag{9}$$

In general, these three scales do not have to be of the same order

$$\mathrm{O}(\Delta_1) \neq \mathrm{O}(\Delta_2) \neq \mathrm{O}(\frac{1}{\Lambda}). \tag{10}$$

Especially, the variation of the wave vector does not have to be as small along the ray path, as often is believed and assumed.

Expanding now $\phi(\mathbf{X}'), \mathbf{E}(\mathbf{X}')$ and $\mathbf{H}(\mathbf{X}')$ around the point $\mathbf{X}$, we obtain

$$\mathbf{E}(\mathbf{X}') = e^{i\phi(\mathbf{X})} e^{i(\mathbf{X}'-\mathbf{X})\cdot\mathbf{K}(\mathbf{X})} \Big\{ \mathbf{E}(;\mathbf{X}) + (\mathbf{X}' - \mathbf{X}) \cdot \frac{\partial}{\partial \mathbf{X}} \mathbf{E}(;\mathbf{X}) + \tag{11}$$
$$\frac{i}{2} \left[(\mathbf{X}' - \mathbf{X})(\mathbf{X}' - \mathbf{X}) : \frac{\partial}{\partial \mathbf{X}} \mathbf{K} \right] \mathbf{E}(;\mathbf{X}) + \mathrm{O}({\Delta_1}^2 \Delta_2) + \mathrm{O}(\Lambda \Delta^3 {\Delta_2}^2) \Big\},$$

where we have used dyadic and double scalar products. A similar expression can be obtained for $\mathbf{H}(\mathbf{X}')$. Fourier transforming the fast varying part of $\sigma^{\text{ker}}(\mathbf{X}-\mathbf{X}';\mathbf{X}), \varepsilon^{\text{ker}}(\mathbf{X}-\mathbf{X}';\mathbf{X})$ and $\mu^{\text{ker}}(\mathbf{X}-\mathbf{X}';\mathbf{X})$ in (1) and using the features of the delta function $\delta(\mathbf{K}'-\mathbf{K})$ and it's derivatives, we can transform (1) into (Suchy & Sabzevari 1992a, Kravtsov 1969)

$$\begin{aligned}
\mathbf{j}(\mathbf{X}) &= e^{i\phi(\mathbf{X})}[\sigma(\mathbf{K};\mathbf{X})(1-i\overleftrightarrow{\partial}-\frac{i}{2}\overleftarrow{\partial})]\cdot\mathbf{E}(;\mathbf{X}) \\
\mathbf{D}(\mathbf{X}) &= e^{i\phi(\mathbf{X})}[\varepsilon(\mathbf{K};\mathbf{X})(1-i\overleftrightarrow{\partial}-\frac{i}{2}\overleftarrow{\partial})]\cdot\mathbf{E}(;\mathbf{X}) \qquad (12)\\
\mathbf{B}(\mathbf{X}) &= e^{i\phi(\mathbf{X})}[\mu(\mathbf{K};\mathbf{X})(1-i\overleftrightarrow{\partial}-\frac{i}{2}\overleftarrow{\partial})]\cdot\mathbf{H}(;\mathbf{X})
\end{aligned}$$

where $\mathrm{O}({\Delta_1}^2\Delta_2)+\mathrm{O}(\Lambda\Delta^3{\Delta_2}^2)\}$ terms have been neglected and the Janus- and Spreading-Operators are defined correspondingly as

$$\overleftrightarrow{\partial}=\frac{\overleftarrow{\partial}}{\partial\mathbf{K}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{X}} \quad \text{and} \quad \overleftarrow{\partial}=\frac{\overleftarrow{\partial^2}}{\partial\mathbf{K}\partial\mathbf{K}}:\frac{\partial}{\partial\mathbf{X}}\mathbf{K}. \qquad (13)$$

To combine them with the Maxwell equations, we first have to calculate the time derivatives of $\mathbf{D}(\mathbf{X})$ and $\mathbf{B}(\mathbf{X})$. To do this, we first write $\mathbf{D}(\mathbf{X})$ (and correspondingly $\mathbf{B}(\mathbf{X})$) in the form (Suchy & Sabzevari 1992 a, Kravtsov 1969)

$$\mathbf{D}=\mathbf{D}(\mathbf{K};\mathbf{X})\exp\{i\phi(\mathbf{X})\} \qquad (14)$$

with

$$\mathbf{D}(\mathbf{K};\mathbf{X})=[\varepsilon(\mathbf{K};\mathbf{X})(1-i\overleftrightarrow{\partial}-\frac{i}{2}\overleftarrow{\partial})]\cdot\mathbf{E}(;\mathbf{X})+\mathrm{O}({\Delta_1}^2\Delta_2)+\mathrm{O}(\Lambda{\Delta_1}^3{\Delta_2}^2) \qquad (15)$$

From (7), we can write

$$\frac{\partial}{\partial\mathbf{K}}=\frac{1}{\Lambda}\frac{\partial}{\partial\mathbf{G}}=\mathrm{O}(\frac{1}{\Lambda}) \qquad (16)$$

$$\frac{\partial^2}{\partial\mathbf{K}\partial\mathbf{K}}=\frac{1}{\Lambda^2}\frac{\partial^2}{\partial\mathbf{G}\partial\mathbf{G}}=\mathrm{O}(\frac{1}{\Lambda^2}), \qquad (17)$$

and from the last two equations follows

$$\frac{\partial}{\partial t}\mathbf{D}(\mathbf{K};\mathbf{X})=\frac{\partial}{\partial t}[\varepsilon(\mathbf{K};\mathbf{X})\cdot\mathbf{E}(;\mathbf{X})]+\mathrm{O}(\frac{\Delta_2}{\Lambda}). \qquad (18)$$

In all equations until now, we kept terms of order $\mathrm{O}(\Delta_1\Delta_2)$, i.e. terms which are products of two small numbers. The last term in (18) is also the product of two small numbers. But we can neglect it compared with the last one, since $\mathrm{O}(1/\Lambda)\ll\mathrm{O}(\Delta_1)$. This follows from the fact that in geometrical optics the scale of the wavelenght and period can be considered as much smaller than the scale of the distance and time, on which different parts of the medium have an essential influence on each other. Using the result (18) and corresponding expressions for the magnetic induction, we finally obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{D}(\mathbf{X}) = \exp\{i\phi(\mathbf{X})\}\varepsilon(\mathbf{K};\mathbf{X})\,\{-i\omega(\mathbf{X})\varepsilon(\mathbf{K};\mathbf{X}) \qquad (19)\\
-\left[\omega(\mathbf{X})\varepsilon(\mathbf{K};\mathbf{X})(\overleftrightarrow{\partial}+\tfrac{1}{2}\overleftarrow{\partial})+\left[\tfrac{\partial}{\partial t}\varepsilon(\mathbf{K};\mathbf{X})\right]_{\mathbf{K}}\right\}\cdot\mathbf{E}(;\mathbf{X})
\end{aligned}$$

and a corresponding expression for $\partial\mathbf{B}(\mathbf{X})/\partial t$, where ε is interchanged with μ and $\mathbf{E}(;\mathbf{X})$ with $\mathbf{H}(;\mathbf{X})$. The time derivatives of $\mathbf{D}(\mathbf{X})$ and $\mathbf{B}(\mathbf{X})$ together with $\mathbf{j}(\mathbf{X})$ from (12) can now be combined with the Maxwell equations (2) to give a system a partial differential equation for the components of the six dimensional electromagnetic field amplitude

$$\mathbf{f}(;\mathbf{X}) = \begin{bmatrix} \mathbf{E}(;\mathbf{X}) \\ \mathbf{H}(;\mathbf{X}) \end{bmatrix}. \tag{20}$$

Since we have different modes propagating in the medium, we first expand the electromagnetic field into these modes

$$\mathbf{f}(\mathbf{X}) = \sum_\alpha e^{\phi_\alpha(\mathbf{X})}\,\mathbf{f}_\alpha(;\mathbf{X}) \tag{21}$$

where $\phi_\alpha(\mathbf{X})$ is the eikonal and $\mathbf{f}_\alpha(;\mathbf{X})$ the electromagnetic amplitude for each mode and each mode has a separate wave vector

$$\mathbf{K}_\alpha(\mathbf{X}) = \frac{\partial\phi_\alpha(\mathbf{X})}{\partial\mathbf{X}}. \tag{22}$$

Then the system of partial differential equations for the electromagnetic amplitudes of the modes can be represented as

$$\sum_\alpha \exp\{i\phi_\alpha(\mathbf{X})\}\left\{\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})(i + \overleftrightarrow{\partial}_\alpha + \frac{1}{2}\overleftarrow{\partial}_\alpha) - \left[\frac{\partial}{\partial t}\mathbf{C}(\mathbf{K}_\alpha;\mathbf{X})\right]_{\mathbf{K}_\alpha}\right\}\cdot\mathbf{f}_\alpha(;\mathbf{X}) = 0 \tag{23}$$

which we call the Eikonal-Maxwell system (Suchy & Sabzevari 1992a), with the Maxwell tensor

$$\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X}) = \begin{pmatrix} \omega_\alpha(\mathbf{k};\mathbf{X})\,\varepsilon(\mathbf{K}_\alpha;\mathbf{X}) + i\,\sigma(\mathbf{K}_\alpha;\mathbf{X}) & \mathbf{k}\times\mathbf{I} \\ & \\ -\mathbf{k}\times\mathbf{I} & \omega_\alpha(\mathbf{k};\mathbf{X})\,\mu(\mathbf{K}_\alpha;\mathbf{X}) \end{pmatrix}, \tag{24}$$

where $\mathbf{I}$ is the unit tensor and $\mathbf{k}$ is the three dimensional spatial part of the wave vector (22) and ω_α the frequency [1]. $\mathbf{C}(\mathbf{K}_\alpha;\mathbf{X})$ is the material tensor

$$\mathbf{C}(\mathbf{K}_\alpha;\mathbf{X}) = \begin{pmatrix} \varepsilon(\mathbf{K}_\alpha;\mathbf{X}) & 0 \\ & \\ 0 & \mu(\mathbf{K}_\alpha;\mathbf{X}) \end{pmatrix}. \tag{25}$$

$\overleftrightarrow{\partial}_\alpha$ and $\overleftarrow{\partial}_\alpha$ are the operators (13) corresponding to each mode. The derivatives to each mode are defined as

$$\frac{\partial}{\partial\mathbf{K}_\alpha}\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X}) = \frac{\partial}{\partial\mathbf{K}}\mathbf{M}(\mathbf{K};\mathbf{X})\Big|_{\mathbf{K}=\mathbf{K}_\alpha} \tag{26}$$

[1] In $\mathbf{K}_\alpha$, it is sufficent to denote the index α to one component only, because of the dispersion relation (30).

3 The eigenvalue spectrum of the modes and polarization vectors

It can be shown (Suchy & Sbzevari 1992a) that the Maxwell-Tensor (24) can be written as the sum of two parts

$$\mathbf{M}(\mathbf{K};\mathbf{X}) = \mathbf{M}_T(\mathbf{K}_T;\mathbf{X}) - \lambda\,\mathbf{M}_L(\mathbf{K}_T;\mathbf{X}) \tag{27}$$

where we habe devided the wave-vector into a longitudinal and transversal part

$$\mathbf{K} = \mathbf{K_T} + \lambda\,\mathbf{g}^\lambda \tag{28}$$

where $\mathbf{g}^\lambda$ is the axis in the direction of the longitudinal part. For example, λ can be chosen as minus the frequency ω. Then $\mathbf{K}_T$ is the spatial wave vector $\mathbf{k}$ and we call these frequency modes. For the case of stratified (not necessarily plane) media λ can be chosen as the component of $\mathbf{k}$ perpendicular to the stratification surfaces. For a thorough discussion of various possibilities see Suchy & Sabzevari 1992a. It should be noted, that using the method represented here, restricts the dispersion of ϵ, σ and μ to the three components of $\mathbf{K}_T$.

The form (27) of the Maxwell tensor is called a generalized characteristic matrix. We can define right and left eigenvectors

$$\begin{aligned} \mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})\cdot\mathbf{g}_\alpha &= [\mathbf{M}_T(\mathbf{K}_T;\mathbf{X}) - \lambda_\alpha(\mathbf{K}_T;\mathbf{X})\,\mathbf{M}_L(\mathbf{K}_T;\mathbf{X})]\cdot\mathbf{g}_\alpha = 0 \\ \bar{\mathbf{g}}_\alpha\cdot\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X}) &= \bar{\mathbf{g}}_\alpha\cdot[\mathbf{M}_T(\mathbf{K}_T;\mathbf{X}) - \lambda_\alpha(\mathbf{K}_T;\mathbf{X})\,\mathbf{M}_L(\mathbf{K}_T;\mathbf{X})] = 0. \end{aligned} \tag{29}$$

Equations (29) have nontrivial solutions only when

$$\mathrm{Det}\,\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X}) = 0. \tag{30}$$

This gives us the dispersion relation. We choose the $\mathbf{g}_\alpha$ as the polarization vectors. The advantage of this choose is, that the right eigenvectors are parallel to the electromagnetic field in a homogeneous medium, as can be seen immediately from (23). We can then devide the space-time into small cells and in each cell the medium can be considered as homogeneous. When we go from one cell to the other, the $\mathbf{g}_\alpha$ will change direction and size. So the first equation of (29) gives us the direction (but not the amount) of the wave field at each space-time point. The left eigenvectors have in general no real physical meaning. They are only useful for further calculations. But they can be considered as waves in a concomitant space (Suchy & Altman 1975).

It is easy to prove the following biorthogornality relation

$$\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\beta = \delta_{\alpha\beta}\,\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\alpha. \tag{31}$$

The right and left eigenvectors are identical when the Maxwell tensor is equal to it's own transpose. In the case of a hermitean Maxwell tensor, the left eigenvectors become the complex conjugate of the right eigenvectors.

Since the right (left) eigenvectors can be defined as an arbitrary linear combination of the columns (rows) of the adjoint of $\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})$, we can define

$$\begin{aligned} \mathbf{g}_\alpha &= \frac{\mathrm{Adj}\,\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})\cdot\mathbf{c}}{\sqrt{\bar{\mathbf{c}}\cdot\mathrm{Adj}\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})\cdot\mathbf{c}}} \\ \bar{\mathbf{g}}_\alpha &= \frac{\bar{\mathbf{c}}\cdot\mathrm{Adj}\,\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})}{\sqrt{\bar{\mathbf{c}}\cdot\mathrm{Adj}\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})\cdot\mathbf{c}}} \end{aligned} \tag{32}$$

where $\bar{\mathbf{c}}$ and $\mathbf{c}$ are arbitrary constant column and row vectors. An expression for the biorthogonality relation (31) can be obtained (Suchy & Sabzevari 1992b)

$$\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\alpha = \Theta\prod_{\gamma\neq\alpha}(\lambda_\alpha-\lambda_\gamma). \tag{33}$$

The factor $\Theta = \Theta(\mathbf{K}_T;\mathbf{X})$ occures because we have a generalized characteristic matrix, i.e. the matrix (27). With this expression, it is possible to normalize the polarization vectors in a suitable form

$$\begin{aligned} \widehat{\mathbf{f}}_\alpha &= \frac{\mathbf{g}_\alpha}{\sqrt{\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\alpha}} = \frac{\mathbf{g}_\alpha}{\sqrt{\Theta\prod_{\gamma\neq\alpha}(\lambda_\alpha-\lambda_\gamma)}} \\ \widehat{\bar{\mathbf{f}}}_\alpha &= \frac{\bar{\mathbf{g}}_\alpha}{\sqrt{\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\alpha}} = \frac{\bar{\mathbf{g}}_\alpha}{\sqrt{\Theta\prod_{\gamma\neq\alpha}(\lambda_\alpha-\lambda_\gamma)}} \end{aligned} \tag{34}$$

and from the biorthogonality relations (31) follow biorthonormality relations

$$\begin{cases} \widehat{\bar{\mathbf{f}}}_\alpha\cdot\mathbf{M}_L\cdot\widehat{\mathbf{f}}_\alpha = 1 & \text{always} \\ \\ \widehat{\bar{\mathbf{f}}}_\alpha\cdot\mathbf{M}_L\cdot\widehat{\mathbf{f}}_\beta = 0 & \lambda_\alpha\neq\lambda_\beta. \end{cases} \tag{35}$$

We define now the complete electromagnetic wave amplitude for each mode by

$$\mathbf{f}_\alpha = a_\alpha(;\mathbf{X})\,\widehat{\mathbf{f}}_\alpha(\mathbf{K}_\alpha;\mathbf{X}) \tag{36}$$

where a_α is the scalar amplitude of the wave. To write the modes in the form of the last equation is of great advantage. The reason is, that $\bar{\mathbf{f}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{f}_\alpha$ is equal to the energy of the wave propagating through the medium (Suchy & Sabzevari 1992a) when collisions are neglected, i.e. hermitean Maxwell tensor. Hence, because of the first of the biorthonormality rations in (35), $|a_\alpha|^2$ is identical to the wave energy of the mode α. In the case of non-hermitean Maxwell tensor, this last quantity is proportional to the wave energy at each space-time point, hence $|a_\alpha|^2/|a_\beta|^2$ gives us directly transmission-, reflection- or absortbtion- coefficients of different modes.

4 Mode coupling point

Since the λ's are functions of space and time, there are some points $\mathbf{X}_0$, where $\lambda_\alpha(\mathbf{K}_T(\mathbf{X}_0);\mathbf{X}_0) = \lambda_\beta(\mathbf{K}_T(\mathbf{X}_0);\mathbf{X}_0)$. Then from (29) and (34) the polarization vectors $\widehat{\mathbf{f}}_\alpha$ and $\widehat{\mathbf{f}}_\beta$ become parallel when the rank of the matrix $\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X})\,(=\mathbf{M}(\mathbf{K}_\beta;\mathbf{X}))$ is equal to 5. This is because if the rank of a matrix is equal to the dimension minus the number of linearly independent eigencolumns of the matrix, then we have only one linearly independent $\widehat{\mathbf{f}}_\alpha\,(=\widehat{\mathbf{f}}_\beta)$. Otherwise, there would be more than one linearly independent eigencolumns for each mode. For example, if the rank would be 4, then we would have two linearly independent eigencolumns for each α and β, i. e. $\mathbf{f}_{\alpha 1}\,(=\mathbf{f}_{\beta 1})$ and $\mathbf{f}_{\alpha 2}\,(=\mathbf{f}_{\beta 2})$. Hence $\mathbf{f}_{\alpha 1}$ would be linearly independent of $\mathbf{f}_{\beta 2}$. In this case, at the point $\mathbf{X}_0$, we can find two vectors $\widehat{\mathbf{f}}_\alpha$ and $\widehat{\mathbf{f}}_\beta$, corresponding to different modes, which are linearly independent, i.e. not parallel. No mode coupling would occure in this case. The two modes would pass by without any influence on each other (Budden 1985, chs. 17.6 and 17.7, Budden & Smith 1974). Hence the exact definition of a coupling point is

$$\text{Coupling point} \quad \Leftrightarrow \quad \{\lambda_\alpha = \lambda_\beta \ \text{ for } \ \mathbf{X} = \mathbf{X}_0 \ \text{ and } \ \mathrm{Rank}\,\mathbf{M}(\mathbf{K}_\alpha;\mathbf{X}) = \mathrm{Rank}\,\mathbf{M}(\mathbf{K}_\beta;\mathbf{X}) = 5\}\,. \tag{37}$$

It should be noted, that at the coupling points the scalar amplitudes a_α and a_β are generally not equal. Physically, coupling does not occure at one mathematical point. Actually there is a small area around the coupling point, where coupling takes place. This coupling area can be defined as

$$\text{Coupling area} \quad \Leftrightarrow \quad \{|\mathbf{X}-\mathbf{X}_0| < \epsilon \iff |\lambda_\alpha - \lambda_\beta| < \delta \ \mid \ \epsilon, \delta \ll 1, \ \text{and} \ \ \lambda_\alpha(\mathbf{X}_0) = \lambda_\beta(\mathbf{X}_0)\}\,. \tag{38}$$

It is possible to estimate the size of ϵ and δ [2]. When two waves reach the border of this area, the coupling is weak. The more they get into the area, the stronger the coupling will be. At the coupling point, the coupling is maximum. Usually, a singularity is to be expected there (for the example of a stratified medium see Sabzevari 1992, 1993, Budden 1985, ch. 16.3, Budden 1972)

From (34) and (35) follow the biorthonormality relations

$$\begin{aligned} \widehat{\widehat{\mathbf{f}}}_\alpha \cdot \mathbf{M}_L \cdot \widehat{\mathbf{f}}_\beta = 1 \quad &\text{for} \quad \mathbf{X} = \mathbf{X}_0 \\ \widehat{\widehat{\mathbf{f}}}_\alpha \cdot \mathbf{M}_L \cdot \widehat{\mathbf{f}}_\beta = 0 \quad &\text{for} \quad \mathbf{X} \neq \mathbf{X}_0. \end{aligned} \tag{39}$$

As we can see from (39), the point $\mathbf{X}_0$ is a "jump singularity", because the scalar product is discontinuos there.

[2] work is in progress and will be represented in the framework of an arbitrary number of waves coupling to each other in a separate paper

5 Transport equations for the amplitude

Since the polarization vectors $\widehat{\mathbf{f}}_\alpha$ can be calculated by the local eigenvector equation (29) at each space-time point $\mathbf{X}$, we need another equation to calculate the scalar amplitudes a_α along the rays to obtain the complete wavefield (36) at each space-time point. Putting (36) into the system (23), a system of partial differential equations for the amplitudes can be obtained (for the technical details of the calculations see Suchy & Sabzevari 1992a)

$$\sum_\gamma e^{\phi_\gamma(\mathbf{X})}\left[\mathbf{W}_{\rho\gamma}\cdot\frac{\partial a_\gamma}{\partial\mathbf{X}}+\Gamma_{\rho\gamma}a_\gamma\right]=0. \tag{40}$$

The transport vectors $\mathbf{W}_{\rho\gamma}$ give the change of the amplitude in the direction of this vector, as can be seen from the last equation itself. They have the form

$$\mathbf{W}_{\rho\gamma}=\delta_{\rho\gamma}\frac{d\mathbf{X}}{d\tau_\gamma}+(\lambda_\rho-\lambda_\gamma)\,\widehat{\overline{\mathbf{f}}}_\rho\cdot\mathbf{M}_L\cdot(\widehat{\mathbf{f}}_\gamma\,\frac{\overleftarrow{d}}{d\mathbf{K}_T}), \tag{41}$$

and the coefficients of the system are

$$\Gamma_{\rho\gamma}=\delta_{\rho\gamma}\frac{\overrightarrow{\partial}_\gamma\,\mathbf{D}_\gamma}{2}+(\lambda_\rho-\lambda_\gamma)\,\widehat{\overline{\mathbf{f}}}_\rho\cdot\mathbf{M}_L\cdot\left(\widehat{\mathbf{f}}_\gamma\,\frac{\overleftarrow{d}}{d\mathbf{K}_T d\mathbf{K}_T}\;:\;\frac{\partial}{\partial\mathbf{X}_T}\mathbf{K}_T\right)+\hat{\Gamma}_{\rho\gamma} \tag{42}$$

with

$$\begin{aligned}\hat{\Gamma}_{\rho\gamma}\;=\;&\widehat{\overline{\mathbf{f}}}_\rho\cdot\Bigg[\mathbf{M}_L\cdot\frac{d\widehat{\mathbf{f}}_\gamma}{d\tau_\gamma}+\left(\frac{d\mathbf{M}_L}{d\tau_\gamma}\right)_{\mathbf{X}}\cdot\widehat{\mathbf{f}}_\gamma\\ &-\mathbf{M}_\gamma\,\frac{\overleftarrow{d}}{d\mathbf{K}_T}\;:\;\left(\frac{\partial}{\partial\mathbf{X}_T}\widehat{\mathbf{f}}_\gamma\right)_{\mathbf{K}_T}+\left(\frac{\partial\mathbf{C}}{\partial t}\right)_{\mathbf{K}_\gamma}\cdot\widehat{\mathbf{f}}_\gamma\Bigg].\end{aligned} \tag{43}$$

In these relations $\mathbf{X}=\mathbf{X}_T+\mathbf{g}_\lambda\mathbf{X}_L$ with $\mathbf{g}_\lambda\cdot\mathbf{g}^\lambda=1$ (compare with (28)). τ_γ is the parametrization of the ray γ and is defined via the Hamiltonian equations (Suchy & Sabzevari 1992a, Berstein & Friedland 1983)

$$\begin{aligned}\tfrac{d\mathbf{X}}{d\tau_\gamma}\;&=\;\tfrac{\partial D_\gamma(\mathbf{K};\mathbf{X})}{\partial\mathbf{K}}\\ \tfrac{d\mathbf{K}}{d\tau_\gamma}\;&=\;-\left[\tfrac{\partial D_\gamma(\mathbf{K};\mathbf{X})}{\partial\mathbf{X}}\right]_{\mathbf{K}},\end{aligned} \tag{44}$$

where D_γ is defined as

$$\mathrm{Det}\,\mathbf{M}(\mathbf{K};\mathbf{X})=-\Theta(\mathbf{K}_T;\mathbf{X})\prod_\rho D_\gamma(\mathbf{K};\mathbf{X}) \tag{45}$$

with

$$D_\gamma(\mathbf{K};\mathbf{X})=\lambda-\lambda_\gamma(\mathbf{K}_T;\mathbf{X}). \tag{46}$$

In the case where all the modes are propagating independently, i.e. when there is no mode coupling, there are no points where the ray paths coincide and the paths are independent from each other. In this case the functions

$$\phi_\gamma(\mathbf{X}) = \int_{\text{ray path}}^{\mathbf{X}} d\mathbf{X}' \cdot \mathbf{K}_\gamma(\mathbf{X}') \tag{47}$$

are independent from each other. The second terms in (41) and (42) become identical to zero and the Kronecker symbol becomes one. From (40) follows then an ordinary differential equation for the scalar amplitude of each mode α along it's ray path

$$\frac{da_\alpha}{d\tau_\alpha} + \Gamma_{\alpha\alpha}\, a_\alpha = 0 \tag{48}$$

with

$$\Gamma_{\alpha\alpha} = \frac{\vec{\partial}_\alpha\, D_\alpha}{2} + \hat{\Gamma}_{\alpha\alpha}. \tag{49}$$

The solution is

$$a_\alpha(\tau_\alpha) = \text{const}\; \exp\left\{-\int^{\tau_\alpha} d\tau'_\alpha\, \Gamma_{\alpha\alpha}\right\} \tag{50}$$

which gives us the amplitude along it's path. For more discussions about the solutions of independent modes see Suchy & Sabzevari 1992a, sec. 7, Sabzevari 1992, 1993.

Now the advantage of the representation (36) (except that it is four-dimensional in space and time) becomes more transparent, since with the last equations, we can calculate the change of the scalar amplitude, and hence the energy directly, without bothering about polarization vectors

6 Coupled wave equations

In this section, we explicitly derive coupled ordinary differential equations for the amplitudes of coupled waves near their coupling point (except the coupling point itself, because there is a jump singularity, but we can approach the coupling point infinitly), i. e. where $\lambda_\alpha \approx \lambda_\beta$.

In the case of the coupling of the two modes α and β, we can see from (47) that ϕ_α and ϕ_β can not be considered as independent anymore, because their ray path coincide in $\mathbf{X}$- and $\mathbf{K}$-space at the coupling point $\mathbf{X}_0$. Since an arbitrary constant phase can be added to the eikonals, we can write near the coupling point in $\mathbf{K}$-space

$$\phi_\alpha(\mathbf{X}) = \int_{\mathbf{X}_0}^{\mathbf{X}} d\mathbf{X}' \cdot \mathbf{K}_\alpha(\mathbf{X}') \approx \int_{\mathbf{X}_0}^{\mathbf{X}} d\mathbf{X}' \cdot \mathbf{K}_\beta(\mathbf{X}') = \phi_\beta(\mathbf{X}). \tag{51}$$

As far as only the two modes couple and are decoupled of the other modes, ϕ_α and ϕ_β are independent of the eikonals corresponding to the other modes. From (40) follows then

$$
\begin{aligned}
\mathbf{W}_{\alpha\alpha}\cdot\tfrac{\partial a_\alpha}{\partial \mathbf{X}}+\mathbf{W}_{\alpha\beta}\cdot\tfrac{\partial a_\beta}{\partial \mathbf{X}}+\Gamma_{\alpha\alpha}a_\alpha+\Gamma_{\alpha\beta}a_\beta &= 0\\
\mathbf{W}_{\beta\alpha}\cdot\tfrac{\partial a_\alpha}{\partial \mathbf{X}}+\mathbf{W}_{\beta\beta}\cdot\tfrac{\partial a_\beta}{\partial \mathbf{X}}+\Gamma_{\beta\alpha}a_\alpha+\Gamma_{\beta\beta}a_\beta &= 0.
\end{aligned}
\tag{52}
$$

For the transport vectors follows from (41)

$$
\mathbf{W}_{\alpha\beta}=(\lambda_\alpha-\lambda_\beta)\,\widehat{\bar{\mathbf{f}}}_\alpha\cdot\mathbf{M}_L\cdot\left(\widehat{\mathbf{f}}_\beta\frac{\overleftarrow{d}}{d\mathbf{K}_T}\right). \tag{53}
$$

It may seem at the first moment, that this term is small, because $\lambda_\alpha\approx\lambda_\beta$. But this assumption is invalid, as we will see soon. To calculate the transport vector, we use (33) to write the equations (34) as

$$
\begin{aligned}
\widehat{\mathbf{f}}_\beta &= (\lambda_\beta-\lambda_\alpha)^{-\frac{1}{2}}\,\frac{\mathbf{g}_\beta}{\sqrt{\Theta\prod_{\gamma\neq\alpha,\beta}(\lambda_\beta-\lambda_\gamma)}}\\
\widehat{\bar{\mathbf{f}}}_\alpha &= (\lambda_\alpha-\lambda_\beta)^{-\frac{1}{2}}\,\frac{\bar{\mathbf{g}}_\alpha}{\sqrt{\Theta\prod_{\gamma\neq\alpha,\beta}(\lambda_\alpha-\lambda_\gamma)}}
\end{aligned}
\tag{54}
$$

From (31) and (33) it is clear that

$$
\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\beta=0\quad\text{for all }\ \lambda_\alpha\text{ and }\lambda_\beta \tag{55}
$$

We differentiat $\widehat{\mathbf{f}}_\beta$ to $\mathbf{K}_T$ and multiply it from the right with $\widehat{\bar{\mathbf{f}}}_\alpha\cdot\mathbf{M}_L(\mathbf{K}_T;\mathbf{X})$ to obtain with (55) for (53)

$$
\mathbf{W}_{\alpha\beta}\;=\;i\,\frac{\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\left(\mathbf{g}_\beta\frac{\overleftarrow{d}}{d\mathbf{K}_T}\right)}{\Theta\sqrt{\prod_{\gamma\neq\alpha,\beta}(\lambda_\alpha-\lambda_\gamma)(\lambda_\beta-\lambda_\gamma)}}. \tag{56}
$$

Since $\mathbf{g}_\beta$ is an arbitrary linear combination of the columns of $\mathrm{Adj}\mathbf{M}(\mathbf{K}_\beta;\mathbf{X})$, the elements of this column are polynomials of λ_β. Hence

$$
\mathbf{g}_\beta\frac{\overleftarrow{d}}{d\mathbf{K}_T}\;=\;\left(\frac{\partial}{\partial\lambda_\beta}\mathbf{g}_\beta\right)\left(\frac{\partial\lambda_\beta}{\partial\mathbf{K}_T}\right)\;+\;\left(\mathbf{g}_\beta\frac{\overleftarrow{\partial}}{\partial\mathbf{K}_T}\right)_{\lambda_\beta}. \tag{57}
$$

It can be shown (see appendix) that near the coupling point in general

$$
\frac{\partial\lambda_\beta}{\partial\mathbf{K}_T}\;\propto\;\frac{1}{\lambda_\alpha-\lambda_\beta}. \tag{58}
$$

This relation is not only valid for the derivatives to $\mathbf{K}_T$, but for the derivatives of all kind of variables. The coefficients of the polynomials in $\mathbf{g}_\beta$ can in general be assumed as analytical, since they depend only on the elements of the matrices $\mathbf{M}_T(\mathbf{K}_T;\mathbf{X})$ and $\mathbf{M}_L(\mathbf{K}_T;\mathbf{X})$, which depend only on the $\epsilon,\mu,\sigma,\mathbf{K}_T$ and $\mathbf{X}$. Therefore the first term in (57) in general is growing relativ to the second as we approach the coupling point. Hence, we can write

$$
\mathbf{g}_\beta\frac{\overleftarrow{d}}{d\mathbf{K}_T}\approx\left(\frac{\partial}{\partial\lambda_\beta}\mathbf{g}_\beta\right)\left(\frac{\partial\lambda_\beta}{\partial\mathbf{K}_T}\right). \tag{59}
$$

Now from the first Hamiltonian equation (44) and the definition of D_β from (46) follows

$$\frac{d\mathbf{X}}{d\tau_\beta} = -\frac{\partial\lambda_\beta}{\partial\mathbf{K}_T} + \mathbf{g}_\lambda \tag{60}$$

Near the coupling point (58) can be used to write (60) as

$$\frac{d\mathbf{X}}{d\tau_\beta} \approx -\frac{\partial\lambda_\beta}{\partial\mathbf{K}_T}. \tag{61}$$

Putting this in (59), it follows from (56)

$$\begin{array}{ll} \mathbf{W}_{\alpha\beta}\cdot\frac{\partial a_\beta}{\partial\mathbf{X}} = \Upsilon_1\frac{da_\beta}{d\tau_\beta} & \mathbf{W}_{\alpha\alpha}\cdot\frac{\partial}{\partial\mathbf{X}} = \frac{d}{d\tau_\alpha} \\ \mathbf{W}_{\beta\alpha}\cdot\frac{\partial a_\alpha}{\partial\mathbf{X}} = \Upsilon_2\frac{da_\alpha}{d\tau_\alpha} & \mathbf{W}_{\beta\beta}\cdot\frac{\partial}{\partial\mathbf{X}} = \frac{d}{d\tau_\beta} \end{array} \tag{62}$$

where

$$\begin{array}{rcccl} \Upsilon_1 & = & \frac{(\lambda_\alpha-\lambda_\beta)\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\left(\frac{\partial}{\partial\lambda_\beta}\mathbf{g}_\beta\right)}{\sqrt{\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\alpha}\sqrt{\bar{\mathbf{g}}_\beta\cdot\mathbf{M}_L\cdot\mathbf{g}_\beta}} & = & -i\frac{\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\left(\frac{\partial}{\partial\lambda_\beta}\mathbf{g}_\beta\right)}{\Theta\sqrt{\prod_{\gamma\neq\alpha,\beta}(\lambda_\alpha-\lambda_\gamma)(\lambda_\beta-\lambda_\gamma)}} \\ \Upsilon_2 & = & \frac{-(\lambda_\alpha-\lambda_\beta)\bar{\mathbf{g}}_\beta\cdot\mathbf{M}_L\cdot\left(\frac{\partial}{\partial\lambda_\alpha}\mathbf{g}_\alpha\right)}{\sqrt{\bar{\mathbf{g}}_\alpha\cdot\mathbf{M}_L\cdot\mathbf{g}_\alpha}\sqrt{\bar{\mathbf{g}}_\beta\cdot\mathbf{M}_L\cdot\mathbf{g}_\beta}} & = & +i\frac{\bar{\mathbf{g}}_\beta\cdot\mathbf{M}_L\cdot\left(\frac{\partial}{\partial\lambda_\alpha}\mathbf{g}_\alpha\right)}{\Theta\sqrt{\prod_{\gamma\neq\alpha,\beta}(\lambda_\alpha-\lambda_\gamma)(\lambda_\beta-\lambda_\gamma)}} \end{array} \tag{63}$$

for $\lambda_\alpha \approx \lambda_\beta$ and $\lambda_\alpha > \lambda_\beta$ (when $\lambda_\alpha, < \lambda_\beta$, the sign of Υ_1 and Υ_2 would be interchanged). When only the two modes α and β couple, whilst the others are decoupled, Υ_1 and Υ_2 are finite. In contrast, by the same methods represented here, it can be shown that the $\Gamma_{\alpha\beta}$, $\Gamma_{\beta\alpha}$, $\Gamma_{\alpha\alpha}$ and $\Gamma_{\beta\beta}$ are growing to infinity as we approach the coupling point. This is due to the second derivative of the second term on the right hand side of (42). From (58) this second derivative is of order $\mathrm{O}(\lambda_\alpha - \lambda_\beta)^{-2}$ and the singularity can not be removed when multiplied by $(\lambda_\alpha - \lambda_\beta)$.

We are now ready to construct the coupled equations. Near the coupling points the paths almost coincide and we can write $d\tau_\alpha \approx d\tau_\beta = d\tau$. Putting (62) in (52), we obtain after some subtractions and additions the system

$$\frac{d}{d\tau}\mathbf{a} - \mathbf{\Gamma}\cdot\mathbf{a} = 0 \tag{64}$$

where

$$\mathbf{a} = \begin{pmatrix} a_\alpha \\ a_\beta \end{pmatrix} \qquad \mathbf{\Gamma} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \tag{65}$$

with

$$\begin{array}{ll} \Gamma_{11} = \frac{\Upsilon_1\Gamma_{\beta\alpha}-\Gamma_{\alpha\alpha}}{1-\Upsilon_1\Upsilon_2} & \Gamma_{12} = \frac{\Upsilon_1\Gamma_{\beta\beta}-\Gamma_{\alpha\beta}}{1-\Upsilon_1\Upsilon_2} \\ \Gamma_{21} = \frac{\Upsilon_2\Gamma_{\alpha\alpha}-\Gamma_{\beta\alpha}}{1-\Upsilon_1\Upsilon_2} & \Gamma_{22} = \frac{\Upsilon_2\Gamma_{\alpha\beta}-\Gamma_{\beta\beta}}{1-\Upsilon_1\Upsilon_2}. \end{array} \tag{66}$$

The solution of (64) is

$$\mathbf{a} = \exp\left\{\int^{\tau} \mathbf{\Gamma}\, d\tau'\right\} \cdot \mathbf{a}_0 \tag{67}$$

where $\mathbf{a}_0$ is an arbitrary constant column.

For a stratified medium, a different formalism than here, which was constructed for a coupling of an arbitrary number of coupled waves, also resulted in coupled wave equations with singular coefficients (Sabzevari 1992, 1993). But the form of the equation were of a kind, that the singularities first had to be removed in order to solve the problem (see also Budden 1985, ch. 16 and 17, Friedland 1985). This is not the case here in general. An example is given below. We believe, that the origin of these singularities, not only for the present case, but for all other cases studied before, could stem from the linearization of the coupling. Perhaps, these singularities would not appear, if the problem would be reconsidered under the aspect of nonlinearity (Tskhakaya 1996). These topics could be the basis for further investigations. The best method to investigate mode-conversion by the system of equations (64) is to use S-matrix methods. Using this method, there is no need to bother about the singularities at the coupling point. For an example see below.

7 The example of a plane-stratified plasma

To give a simple example of how to handle with the above formulas in practical cases and also to have a cheque of the correctness of the formalism, we consider the well known case of a plane stratified isotropic collisionless cold plasma with an arbitrary number of ion components. The magnetic field is in the z-direction normal to the stratification surfaces. The wave is in the $x-z$-plane and θ the angle between the k-vector and the z-direction. In this case, the Maxwell tensor has the simple form (Swanson 1989, ch. 2.1.2)

$$\begin{bmatrix} \omega\epsilon_0 P & 0 & 0 & 0 & -k\cos\theta & 0 \\ 0 & \omega\epsilon_0 P & 0 & k\cos\theta & 0 & -k\sin\theta \\ 0 & 0 & \omega\epsilon_0 P & 0 & k\sin\theta & 0 \\ 0 & k\cos\theta & 0 & \omega\mu_0 & 0 & 0 \\ -k\cos\theta & 0 & k\sin\theta & 0 & \omega\mu_0 & 0 \\ 0 & -k\sin\theta & 0 & 0 & 0 & \omega\mu_0 \end{bmatrix} \tag{68}$$

where

$$P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}. \tag{69}$$

We consider monochromatic wave-guide modes, i.e. the component of the wave vector in the z-direction. Hence $\lambda = k_z = k\cos\theta$ and from Snell's law $\mathbf{K}_T = k_x = k\sin\theta = (\omega/c_0)\sin\theta_0 =$const.,

where c_0 is the speed of light in vacuum and θ_0 the angle, the wave makes with the z-direction before entering the plasma, i.e. in the vacuum. Then, we have

$$\mathbf{M}_L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{70}$$

The coefficient (42) reduce to

$$\Gamma_{\rho\gamma} = \widehat{\bar{\mathbf{f}}}_\rho \cdot \mathbf{M}_L \cdot \left(\frac{d}{dz} \widehat{\mathbf{f}}_\gamma \right). \tag{71}$$

To derive the right and left eigenvectors $\mathbf{g}$ and $\bar{\mathbf{g}}$, it suffices first to calculate the second row and column of $\operatorname{Adj}\mathbf{M}$ respectivly. We define then

$$\bar{c} = (0,1,0,0,0,0) \qquad \text{and} \qquad c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

and obtain from (32)

$$\mathbf{g}_\alpha = \bar{\mathbf{g}}_\alpha = \frac{\omega^3}{c_0^2}\sqrt{\frac{\mu_0}{\omega}} \begin{bmatrix} 0 \\ (P(P-n_\alpha^2))^{\frac{1}{2}} \\ 0 \\ \sqrt{\frac{\epsilon_0}{\mu_0}}(P(P-n_\alpha^2))^{\frac{1}{2}} n_\alpha \cos\theta \\ 0 \\ \sqrt{\frac{\epsilon_0}{\mu_0}}(P(P-n_\alpha^2))^{\frac{1}{2}} n_\alpha \sin\theta \end{bmatrix} \tag{73}$$

where $n_\alpha = (c_0/\omega)k_\alpha$ is the refractive index. From (34) we obtain

$$\widehat{\mathbf{f}}_\alpha = \widehat{\overline{\mathbf{f}}}_\alpha = \sqrt{\frac{c_0\mu_0}{2}} \begin{bmatrix} 0 \\ \\ (n_\alpha \cos\theta)^{\frac{-1}{2}} \\ \\ 0 \\ \\ \sqrt{\frac{\epsilon_0}{\mu_0}}(n_\alpha \cos\theta)^{\frac{1}{2}} \\ \\ 0 \\ \\ \sqrt{\frac{\epsilon_0}{\mu_0}} n_\alpha^{\frac{1}{2}} \sin\theta (\cos\theta)^{\frac{-1}{2}} \end{bmatrix}. \tag{74}$$

Using (73), the equations (63) can be calculated

$$\Upsilon_{\frac{1}{2}} = \pm \frac{n_\alpha - n_\beta}{2(n_\alpha n_\beta)^{1/2}} \left\{ 1 - \frac{n_{\beta \atop \alpha}^2 + n_\alpha n_\beta}{P(P - n_{\beta \atop \alpha}^2)} \cos^2\theta \right\} \tag{75}$$

and from (74) and (71)

$$\Gamma_{\alpha,\beta} = \frac{1}{4\sqrt{n_\alpha n_\beta}\cos\theta} \left(1 - \frac{n_\alpha}{n_\beta}\right) \frac{d}{dz}(n_\beta \cos\theta). \tag{76}$$

When we consider decoupled modes, i.e. cases where nowhere mode coupling occures, for the last equation follows $\Gamma_{\alpha\alpha} = \Gamma_{\beta\beta} = 0$, and then from (48) follows that the scalar part of the amplitude (but not the polarization vector) does not change when the wave propagates through the medium. This is to be expected, since wave absorbtion is a kinetic effect and does not occure in a cold plasma (see for example Antonson & Manheimer 1978, sec.II), which also immediately can be concluded from the fact that the entropy can not change, when temperature is absent. In a cold plasma, mode coupling can occure only in the form of reflection and transmission, without absorbtion.

We investigat now reflection at a cut-off. There, we have (Swanson 1989, ch. 2.1.2)

$$n_\alpha = -n_\beta = P^{1/2} \to 0. \tag{77}$$

Hence (64) can be used, since (58) is valid. For (75) and (76) follows from (77)

$$\Upsilon_{\frac{1}{2}} = \pm i \left(\frac{\cos^2\theta}{P} - 1 \right) \tag{78}$$

$$\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha} = \frac{i}{2P^{1/2}} \frac{d}{dz} \ln(P^{1/2}\cos\theta) \tag{79}$$

and from (76), (71) and (66) follows for (64)

$$\frac{d}{dz}\left(\begin{array}{c} a_\alpha \\ a_\beta \end{array}\right) + \frac{\Gamma_{\alpha\beta}}{\Upsilon_1^2+1}\left(\begin{array}{cc} \Upsilon_1 & 1 \\ -1 & \Upsilon_1 \end{array}\right)\left(\begin{array}{c} a_\alpha \\ a_\beta \end{array}\right) = 0. \tag{80}$$

Using Snell's law, this can be written in the form of an equation for the difference of the energy of the incoming and reflected wave near the cut-off

$$\frac{d}{dz}\ln\left(|a_\alpha|^2 - |a_\beta|^2\right) = \frac{-P^{3/2}}{2\sin^4\theta_0}\frac{dP}{dz} \quad \text{for} \quad \sin\theta_0 \neq 0 \tag{81}$$

From the definition of P in (69), it is clear that the derivative of P to z is always finite, hence

$$\frac{d}{dz}\ln\left(|a_\alpha|^2 - |a_\beta|^2\right) \to 0 \quad \text{for} \quad P \to 0. \tag{82}$$

(The case $\sin\theta_0 = 0$ can not be calculated by the above method, since in this case Snell's law is everywhere identical to zero. But physically, we can suppose θ_0 to be small and let the limit $P \to 0$ go before $\theta_0 \to 0$. Anyhow, this problem does not occure when calculating the S-matrix, which gives equivalent results, compare eq. (85)).

As to be expected, the difference of the energy of the incoming and reflected wave is conserved at the cut-off, because there no part of the wave is transmitted. This is of course not the case, when we are away from the cut-off, where P is finite. There the difference of the energy of the incoming and reflected wave is not conserved anymore, because at each stratification surface a part of the wave is transmitted toward the cut-off. It is only at the cut-off, where pure reflection takes place.

Solutions for the amplitudes a_α and a_β can be obtained near the coupling point by S-matrix methods (Volland 1962 a, b, Budden 1985, ch. 18.5). We define the cut off point by z_0, i.e. $P(z_0) = 0$. z_2 is a point above, and z_1 a point below z_0. The solutions near z_0 can then be written as

$$\mathbf{a}(z_2) = \mathbf{A}(z_2, z_1) \cdot \mathbf{a}(z_1) \tag{83}$$

where the matrizant is defined as (Gantmacher 1960)

$$\mathbf{A}(z_2, z_1) = \mathbf{I} + \int_{z_1}^{z_2} d\tau_1\, \mathbf{\Gamma}(\tau_1) + \int_{z_1}^{z_2} d\tau_1\, \mathbf{\Gamma}(\tau_1) \cdot \int_{z_1}^{\tau_1} d\tau_2\, \mathbf{\Gamma}(\tau_2) + \cdots. \tag{84}$$

Since z_0 is a cut-off, $a_\alpha(z_2)$ and $a_\beta(z_2)$ are the downgoing and upgoing modes above z_0 respectively and $a_\alpha(z_1)$ and $a_\beta(z_1)$ the upgoing and downgoing modes below z_0 respectively (compare to Budden 1985, eq. 18.20).

For our purpose, it is sufficient to calculate the matrizant up to first order. We obtain

$$\mathbf{A}(z_2,z_1) \;=\; \left[\begin{array}{cc} 1-\frac{P^{5/2}(z_2)-P^{5/2}(z_1)}{10\sin^4\theta_0} & \frac{-i(P^{9/2}(z_2)-P^{9/2}(z_1))}{18\sin^6\theta_0} \\ & \\ \frac{i(P^{9/2}(z_2)-P^{9/2}(z_1))}{18\sin^6\theta_0} & 1-\frac{P^{5/2}(z_2)-P^{5/2}(z_1)}{10\sin^4\theta_0} \end{array}\right] \qquad \text{for} \quad \sin\theta_0 \neq 0$$

$$(85)$$

$$\mathbf{A}(z_2,z_1) \;=\; \left[\begin{array}{cc} 1-\frac{P^{1/2}(z_2)-P^{1/2}(z_1)}{2} & \frac{-i(P^{3/2}(z_2)-P^{3/2}(z_1))}{6} \\ & \\ \frac{i(P^{3/2}(z_2)-P^{3/2}(z_1))}{6} & 1-\frac{P^{1/2}(z_2)-P^{1/2}(z_1)}{2} \end{array}\right] \qquad \text{for} \quad \sin\theta_0 = 0.$$

The diagonal elements of the matrizant are the reflection and the off-diagonal elements the transmission coefficients (Budden 1985, eq. 18.21, Volland 1962 a, eq. 21). We see, that at the cut-off, where $z_1, z_2 \rightarrow z_0$, i.e. $P \rightarrow 0$, the reflection coefficients become 1 and the transmission coefficients 0, as is to be expected.

Appendix

Equation (45) can be written with the help of (46) as

$$\mathrm{Det}\,\mathbf{M}(\mathbf{K};\mathbf{X}) \;=\; \Theta\,(\lambda-\lambda_\alpha)(\lambda-\lambda_\beta)\prod_{\gamma\neq\alpha,\beta}(\lambda-\lambda_\gamma) \;=\; \Theta(\lambda^2+B\lambda+C)\prod_{\gamma\neq\alpha,\beta}(\lambda-\lambda_\gamma) \qquad (86)$$

where $B = B(\mathbf{K}_T;\mathbf{X})$ and $C = C(\mathbf{K}_T;\mathbf{X})$ are analytical. The roots λ_α and λ_β are

$$\lambda_{\substack{\alpha\\ \beta}} = \frac{-B \pm \sqrt{B^2-4C}}{2}. \qquad (87)$$

At the coupling point, we have

$$B^2 - 4C = 0 \qquad (88)$$

Hence the coupling point is a branch point (for the example of a plane stratified medium see Budden 1985, sec. 16.3 or Budden 1972, Budden & Smith 1974). We have

$$\frac{\partial}{\partial\mathbf{K}_T}\lambda_{\substack{\alpha\\ \beta}} = -\frac{1}{2}\frac{\partial B}{\partial\mathbf{K}_T} \pm \frac{1}{4}\left[\frac{\partial}{\partial\mathbf{K}_T}(B^2-4C)\right]\frac{1}{\lambda_\alpha-\lambda_\beta} \qquad (89)$$

from which near the coupling point follows equation (58)

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